

On a difference in topological nature between Cantor middle-third set and Sierpiński carpet

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Abstract

Both Cantor middle-third set and Sierpiński carpet are self-similar, perfect, compact metric spaces. In spite of the similarity of the mathematical procedure of construction, there exists between them a fundamental difference in topological nature, and this difference affects the methods of construction of an interesting non-trivial quotient space of them. The totally disconnectedness (or, more generally, zero-dimensional) enables Cantor middle-third set to have a non-trivial quotient space which is self-similar. On the other hand, concerning Sierpiński carpet, because of the connectedness of its structure, no non-trivial quotient space which is self-similar can be constructed by such an elegant procedure as that for Cantor middle-third set. Various topologically significant nature specific to Cantor middle-third set owe mainly to the totally disconnectedness of the set.

Keywords: self-similar, quotient space, zero-dimensional space, perfect space

1 Introduction

Since the procedure of the block construction or the coarse graining of a space corresponds mathematically to that of the construction of a quotient space (2-D.) of a space, nonlinear science researchers have been interested in quotient spaces[1,2].

In the present paper, between Cantor middle-third set (CMTS) and Sierpiński carpet (SC) (2-C.) both of which are perfect (2-A.), compact and self-similar (2-B.) a difference in topological nature will be made clear through the investigations of the procedure of construction of an interesting quotient space. In fact, owing to the zero-dimensional (0-dim) (2-A.) or totally disconnectedness of CMTS, some interesting non-trivial quotient space of CMTS such as one with a self-similarity can be constructed in an elegant way. Contrary to this, such an interesting non-trivial quotient space is hardly constructed for SC at least in the same elegant procedure with that for CMTS because of the connectedness of SC. In the following sections, this fact will be confirmed.

2 Definitions and Statements

Some definitions and statements which will be used in the section 3 are listed below.

2-A. A topological space (Z, τ) is said to be zero-dimensional (0-dim) provided that at every point $p \in Z$ and for every $U \in \tau$ containing p , there exists $u \in \tau \cap \mathfrak{S}$ (\mathfrak{S}

denotes the set of all closed sets of Z , that is, u is an open and closed set) such that $p \in u \subset U$. In any compact T_2 -space, the notion of 0-dim coincides with that of the totally disconnectedness[3,4] and it is easy to check that any 0-dim T_0 -space is totally disconnected.

A topological space (Z, τ) is said to be perfect provided that a singleton $\{p\}$, $p \in Z$ is not τ -open, that is, any non-empty open set of (Z, τ) has at least two points. Any non-empty open set of a perfect space is perfect as a subspace.

Both 0-dim and perfect are topological properties, that is, 0-dim and perfect are invariant under the homeomorphism[5].

2-B. A topological space (A, τ) is self-similar provided that i) it is metrizable, that is, τ is identical with a metric topology τ_d defined by a metric d on A . ii) there exists a set of contractions $\{f_j : (A, \tau_d) \rightarrow (A, \tau_d); j = 1, \dots, m \ (2 \leq m < \infty)\}$ such that

$$\bigcup_{j=1}^m f_j(A) = A.$$

Especially, we call a metric space (A, τ_d) self-similar provided that the condition ii) is satisfied with respect to this metric d .

A sufficient condition for a topological space to be self-similar is given in the following statement.

Statement. *The existence of a self-similar space which is homeomorphic to (Y, τ) is sufficient for a topological space (Y, τ) to be self-similar.*

(Concerning the proof, see Appendix 1).)

2-C. CMTS is a subspace of closed interval $[0, 1]$ such that $\text{CMTS} = \bigcup_{j=1}^2 f_j(\text{CMTS})$ where

the contractions $f_1(x)$ and $f_2(x)$ are $x/3$ and $x/3 + 2/3$, $x \in [0, 1]$, respectively. The sum of two contraction coefficients is $2/3 < 1$.

Any metric space which is 0-dim, perfect and compact is homeomorphic to CMTS[6].

SC is a subspace of square $[0, 1] \times [0, 1]$ such that $\text{SC} = \bigcup_{j=1}^8 f_j(\text{SC})$ where

$$\begin{aligned} f_1(x_1, x_2) &= (x_1/3, x_2/3), \quad f_2(x_1, x_2) = (x_1/3 + 1/3, x_2/3), \\ f_3(x_1, x_2) &= (x_1/3 + 2/3, x_2/3), \quad f_4(x_1, x_2) = (x_1/3, x_2/3 + 1/3), \quad f_5(x_1, x_2) = (x_1/3 + \\ &2/3, x_2/3 + 1/3), \quad f_6(x_1, x_2) = (x_1/3, x_2/3 + 2/3), \quad f_7(x_1, x_2) = (x_1/3 + 1/3, x_2/3 + \\ &2/3), \quad f_8(x_1, x_2) = (x_1/3 + 2/3, x_2/3 + 2/3), \end{aligned}$$

$(x_1, x_2) \in [0, 1] \times [0, 1]$. The sum of eight contraction coefficients is $8/3 \geq 1$.

In general, the following statement holds.

Statement. *Let (Z, τ_d) be a compact metric space and the system*

$\{f_j : (Z, \tau_d) \rightarrow (Z, \tau_d), j = 1, \dots, m\}$ be a set of contractions

$d(f_j(z), f_j(z')) \leq \alpha_j(\eta)d(z, z')$ for $d(z, z') < \eta$, $0 < \alpha_j(\eta) < 1$, $\inf_{\eta>0} \alpha_j(\eta) > 0$, $j = 1, \dots, m \ (2 \leq m < \infty)$.

Then, there exists a non-empty compact set $X \subset Z$ such that $\bigcup_{j=1}^m f_j(X) = X$. This self-

similar set X is unique in the set of all non-empty compact sets of Z .
 If the system of contractions $\{f_j\}$ satisfies two conditions

i) Each f_j is one to one,

ii) The set $\bigcup_{j=1}^m \{z \in Z; f_j(z) = z\}$ is not a singleton,

then the self-similar compact set X is perfect.

Furthermore, if the system of contractions $\{f_j\}$ satisfies the additional third condition

iii) $\sum_{j=1}^m \inf_{\eta > 0} \alpha_j(\eta) < 1$

the self-similar compact set X is not only perfect but 0-dim.
 (Concerning the proof, see Appendix 2).)

Thus, from this statement, SC is perfect and compact, on the other hand, CMTS is perfect, compact and 0-dim.

2-D. A quotient space of a space X is defined by a classification of all points in X through the identification of the different points based on an equivalence relation. If $f : (X, \tau) \rightarrow (Y, \tau')$ is a continuous, onto, closed map (A map f is said to be a closed map provided that for any closed set K in X , $f(K)$ is a closed set in Y .), then $h : (Y, \tau') \rightarrow (\mathcal{D}_f, \tau(\mathcal{D}_f))$, $y \mapsto f^{-1}(y)$ is a homeomorphism. Here the set \mathcal{D}_f is $\{f^{-1}(y) \subset X; y \in Y\}$ and the topology $\tau(\mathcal{D}_f)$ is $\{\mathcal{U} \subset \mathcal{D}_f; \bigcup \mathcal{U} = \bigcup_{D \in \mathcal{U}} D \in \tau, \text{ each } D \in \mathcal{D}_f \text{ is } f^{-1}(y) \text{ for some } y \in Y\}$. Topological space $(\mathcal{D}_f, \tau(\mathcal{D}_f))$ is a quotient space of (X, τ) based on the equivalence relation \sim

$$x \sim x' \stackrel{\text{def}}{=} f(x) = f(x')$$

Sometimes, the quotient space $(\mathcal{D}_f, \tau(\mathcal{D}_f))$ is called a decomposition space of (X, τ) due to f .

In general, a quotient space $(\mathcal{D}, \tau(\mathcal{D}))$ of (X, τ) where $\mathcal{D} = \{\{x\}; x \in X\}$ ($\{x\}$ denotes a singleton) and $\tau(\mathcal{D}) = \{\mathcal{U} \subset \mathcal{D}; \bigcup \mathcal{U} \in \tau\}$ is called the trivial quotient space. The trivial quotient space \mathcal{D} of X is homeomorphic to X . Therefore, if X is self-similar, from the statement in 2-B., \mathcal{D} must be self-similar.

2-E. Statement. Let (Z, τ) be a 0-dim, perfect T_0 (necessarily T_2)-space. Then, there exists a partition $\{Y, Z - Y\}$ of Z where $Y \in (\tau \cap \mathfrak{S}) - \{\emptyset\}$ and $Y \subsetneq Z$. Here the subspace (Y, τ_Y) is 0-dim and perfect.
 (Concerning the proof, see Appendix 3).)

2-F. [7] Let (X, τ) be a compact normal space and (X_i, τ_{X_i}) , $X_i \in \mathfrak{S} - \{\phi\}$, $i \in \mathbf{N}$ be connected subspaces of (X, τ) such that $X_{i+1} \subset X_i$, $i \in \mathbf{N}$. Then, $E = \bigcap_{i \in \mathbf{N}} X_i$ is not empty and the subspace (E, τ_E) of (X, τ) is compact and connected.

3 Quotient spaces

3.1 Quotient space of CMTS

Since CMTS is a 0-dim, perfect, compact metric space (2-C.), from 2-E., there exists a 0-dim, perfect, compact subspace (Y, τ_{d_Y}) of CMTS such that $Y \subsetneq \text{CMTS}$. For example, $Y = \text{CMTS} \cap [0, 1/3]$. Here the notation d_Y denotes the restriction of the metric d ($d(x, x') = |x - x'|$) on Y . Now, let $f : (\text{CMTS}, \tau_d) \rightarrow (Y, \tau_{d_Y})$ be a not one to one continuous, onto map. The simple one

$$f(x) = \begin{cases} x, & x \in Y \\ q, & x \in \text{CMTS} - Y \end{cases}$$

is an example. Here q is an arbitrarily chosen point of Y . Since the singleton $\{q\}$ is a closed set in a T_1 -space, it is easy to see that the map f is a continuous, closed map from CMTS onto Y . From 2-D., $h : (Y, \tau_{d_Y}) \rightarrow (\mathcal{D}_f, \tau(\mathcal{D}_f)), y \mapsto f^{-1}(y)$ is a homeomorphism. Here, $(\mathcal{D}_f, \tau(\mathcal{D}_f))$ is a quotient space of CMTS due to the above map f . Since (Y, τ_{d_Y}) is a 0-dim, perfect, compact metric space, from 2-C., it is homeomorphic to CMTS. Then, the quotient space \mathcal{D}_f is also homeomorphic to CMTS. According to the statement in 2-B., \mathcal{D}_f is self-similar. It must be noted that this \mathcal{D}_f is not the trivial quotient set $\{\{x\}; x \in \text{CMTS}\}$ because f is not one to one. In fact, for example, if $Y = \text{CMTS} \cap [0, 1/3]$, the set $\{q\} \cup (\text{CMTS} \cap [2/3, 1])$ is a point of \mathcal{D}_f . Namely, there exists a self-similar non-trivial quotient space of CMTS. Furthermore, since \mathcal{D}_f is homeomorphic to CMTS, it is a self-similar, 0-dim, perfect, compact metric space (Concerning the metrizability of \mathcal{D}_f , see Appendix 1). Then, we can obtain a self-similar quotient space of \mathcal{D}_f which is a 0-dim, perfect, compact metric space again. Continuing this process, we attain an infinite sequence $\{\mathcal{D}_i; i = 0, 1, 2, \dots\}$ in which $\mathcal{D}_0 = \text{CMTS}$ and each \mathcal{D}_i is a quotient space of \mathcal{D}_{i-1} .

3.2 Quotient space of SC

According to the statement in 2-C., SC is perfect and compact. CMTS is also perfect and compact, and furthermore, 0-dim. In this paragraph, using 2-F., we will demonstrate that SC is a connected space. SC is given by the intersection $\bigcap_{i \in \mathbf{N}} X_i$ of closed

sets $X_i = \bigcup_{j_1 \dots j_i} f_{j_1} \circ \dots \circ f_{j_i}([0, 1] \times [0, 1])$, $j_1 \in \{1, \dots, 8\}, \dots, j_i \in \{1, \dots, 8\}$, $i \in \mathbf{N}$ in $[0, 1] \times [0, 1]$. Here each X_i has finite number of windows and obeys the relation $X_{i+1} \subset X_i$. Fig.1 shows, for example, X_2 with 9 windows. As shown in Fig.1, any two points p and q are joined by an arc in X_2 . Therefore, X_2 is arcwise connected, that is, X_2 is connected[8]. In the same way, each X_i is convinced to be connected and compact. Applying 2-F. to a

compact normal space $[0, 1] \times [0, 1]$, we can see that non-empty compact set $\bigcap_{i \in \mathbf{N}} X_i = \text{SC}$ is connected. There exists a fundamental difference in topological nature between SC and CMTS. CMTS is 0-dim, i.e, totally disconnected, on the other hand, SC is connected. The procedure of construction of the self-similar quotient space of CMTS is based on the existence of an adequate partition of CMTS and the existence of this partition is based on the 0-dim of CMTS. Thus, concerning SC which is connected, no non-trivial quotient space which for CMTS.

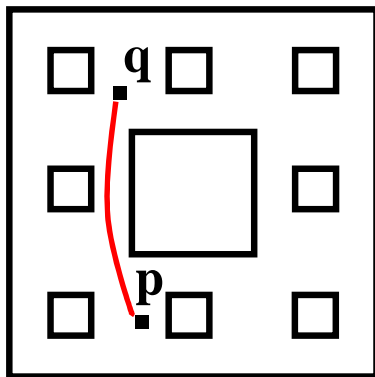


Figure 1: The second step X_2 of the construction of Sierpiński carpet $= \bigcap_{i \in \mathbf{N}} X_i$. Any two points p and q are joined by an arc in X_2 .

4 Conclusion

Results obtained concerning the difference in topological nature between CMTS and SC are as follows.

- 1) A topological space which is homeomorphic to a self-similar space is self-similar. There exists a non-trivial quotient space \mathcal{D} of CMTS which is homeomorphic to CMTS. Since CMTS is self-similar, quotient space \mathcal{D} is also self-similar.
- 2) Above result 1) owes mainly to the 0-dim (the totally disconnectedness) of the structures of CMTS. Therefore, it is impossible for SC which is a connected space to have a self-similar quotient space under the exactly same elegant procedure of construction as that employed for CMTS.

Finally we note that, because of the connectedness of their structures, the same situations with Sierpiński carpet exist for Sierpiński gasket and Sierpiński sponge (Menger sponge)[9].

Appendix

1). Proof of the statement in 2-B.

Let a topological space (X, T) be self-similar and homeomorphic to (Y, τ) . By the above definition of self-similarity, there exists a metric d on X such that $T = \tau_d$ and there exists a system of contractions

$p_j : (X, \tau_d) \rightarrow (X, \tau_d)$, $d(p_j(x), p_j(x')) \leq \alpha_j(\eta)d(x, x')$ for $d(x, x') < \eta$, $0 \leq \alpha_j(\eta) < 1$, $j = 1, \dots, m$ ($2 \leq m < \infty$) satisfying the relation $\bigcup_{j=1}^m p_j(X) = X$. Using a homeomorphism $h : (X, \tau_d) \rightarrow (Y, \tau)$, we can define a metric ρ on Y as

$$\rho(y, y') = d(h^{-1}(y), h^{-1}(y')), \quad y, y' \in Y.$$

The metric topology τ_ρ is identical with the initial topology τ . From the relations 1) and 2) below, $(Y, \tau)(= (Y, \tau_\rho))$ is self-similar based on a system of contractions $q_j : (Y, \tau_\rho) \rightarrow (Y, \tau_\rho)$, $j = 1, \dots, m$ where q_j is topologically conjugate to p_j with the above homeomorphism h , that is, $q_j = h \circ p_j \circ h^{-1}$.

$$\begin{aligned} 1) \quad & \rho(q_j(y), q_j(y')) = d(h^{-1}(q_j(y)), h^{-1}(q_j(y'))) \\ & = d(p_j(h^{-1}(y)), p_j(h^{-1}(y'))) \leq \alpha_j(\eta)d(h^{-1}(y), h^{-1}(y')) \\ & = \alpha_j(\eta)\rho(y, y') \text{ for } \rho(y, y') < \eta. \end{aligned}$$

$$2) \quad \bigcup_{j=1}^m q_j(Y) = \bigcup_{j=1}^m q_j(h(X)) = h\left(\bigcup_{j=1}^m p_j(X)\right) = h(X) = Y. \quad \square$$

2). Proof of the statement in 2-C.

Let $C(Z)$ and d_H be the set of all non-empty compact sets of (Z, τ_d) and the Hausdorff metric, respectively. It is known[10] that the set dynamical system

$$T : (C(Z), \tau_{d_H}) \rightarrow (C(Z), \tau_{d_H}), \quad A \mapsto \bigcup_{j=1}^m f_j(A)$$

has unique fixed point X ; that is, there exists unique non-empty compact set $X(\subset Z)$ such that $\bigcup_{j=1}^m f_j(X) = X$.

The condition *iii*) is known[11] as a condition which makes the set X 0-dim.

To complete the proof, let us show that the conditions *i*) and *ii*) are sufficient for the subspace (X, τ_{d_X}) of (Z, τ_d) (d_X denotes the restriction of the metric d on X) to be perfect. Namely, let us show that any point x of X is an accumulation point of X . Since any contraction on a complete metric space has one, only one, fixed point[10], it is obvious that the condition *ii*) requires that X is not a singleton. Let x_0 and x'_0 be different two points of X . For any point $x \in X$ and for any $\varepsilon > 0$, let us consider an open sphere $S(x, \varepsilon) = \{z \in Z; d(x, z) < \varepsilon\}$. The relation $\bigcup_{j_1 \dots j_k} X_{j_1 \dots j_k} = X$ is valid for any $k \geq 1$ [11].

Here $X_{j_1 \dots j_k}$ denotes $f_{j_1} \circ \dots \circ f_{j_k}(X)$, $j_1 \in \{1, \dots, m\}, \dots, j_k \in \{1, \dots, m\}$, and the diameter of $X_{j_1 \dots j_k} \rightarrow 0$ ($k \rightarrow \infty$)[11]. There exist k and a k -tuple $j_1 \dots j_k$ such that $x \in X_{j_1 \dots j_k}$ and the diameter of $X_{j_1 \dots j_k} < \varepsilon$. Since the contractions f_1, \dots, f_m are all one to one from the condition *i*), the point $p = f_{j_1} \circ \dots \circ f_{j_k}(x_0) \in X_{j_1 \dots j_k} \subset X$ and

the point $p' = f_{j_1} \circ \cdots \circ f_{j_k}(x'_0) \in X_{j_1 \cdots j_k} \subset X$ must be different. Therefore, at least either $S(x, \varepsilon) - \{x\} \ni p$ or $S(x, \varepsilon) - \{x\} \ni p'$ holds. This means that the point x is an accumulation point of X . \square

3). Proof of the statement in 2-E.

Since (Z, τ) is perfect, (Z, τ) has at least two points z and z' . Without loss of generality, there exists $U \in \tau$ such that $z \in U$ and $z' \notin U$. 0-dim guarantees the existence of $u \in \tau \cap \mathfrak{S}$ such that $z \in u \subset U$. Taking this open and closed set u as Y , we obtain a 0-dim, perfect, proper subspace (Y, τ_Y) of (Z, τ) . \square

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References

- [1] S.K.Ma, Modern theory of critical phenomena, Benjamin (1976) p.246.
- [2] A.Fernández, J. Phy. A **21**, L295 (1988).
- [3] A.Illanes and S.B.Nadler Jr., Hyperspace, Marcel Dekker (1999) p.102.
- [4] A.Kitada, Isohkukan to sono ouyo, Asakura shoten (2011) p.39 (Japanese).
- [5] The map $h : (A, \tau) \rightarrow (A', \tau')$ is said to be a homeomorphism provided that h is one to one, onto and continuous together with the inverse function $h^{-1} : (A', \tau') \rightarrow (A, \tau)$. Roughly speaking, "the space A is homeomorphic to the space A' " means that A and A' are geometrically equivalent. Ref.[4] p.24.
- [6] S.B.Nadler Jr., Continuum theory, Marcel Dekker (1992) p.105, p.109.
- [7] S.B.Nadler Jr., Continuum theory, Marcel Dekker (1992) p.6.
- [8] A.Kitada, Isohkukan to sono ouyo, Asakura shoten (2011) p.48 (Japanese).
- [9] M.Matsushita, Furakutal no butsurei(I), shoukabou (2004) p.15 (Japanese).
- [10] S.Nakamura, T.Konishi and A.Kitada, Journal of the Physical Society of Japan **64** (1995) 731.
- [11] A.Kitada, T.Konishi, T.Watanabe, Chaos, Solitons & Fractals **13** (2002) 363.